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Abstract. We consider non-degenerate, finitely supported random walks on a free group. We show that the entropy and the linear drift vary analytically with the probability of constant support.

1. Introduction

Let F be a finitely generated group and for $x \in F$, denote $|x|$ the word length of x. Let p be a finitely supported probability measure on F and define inductively, with $p^{(0)}$ being the Dirac measure at the identity e,

$$p^{(n)}(x) \;=\; [p^{(n-1)} \star p](x) \;=\; \sum_{y \in F} p^{(n-1)}(xy^{-1})p(y).$$

Some of the asymptotic properties of the probabilities $p^{(n)}$ as $n \to \infty$ are reflected in two nonnegative numbers, the entropy h_p and the linear drift ℓ_p:

$$h_p := \lim_n -\frac{1}{n} \sum_{x \in F} p^{(n)}(x) \ln p^{(n)}(x), \quad \ell_p := \lim_n \frac{1}{n} \sum_{x \in F} |x| p^{(n)}(x).$$

Erschler asks whether h_p and ℓ_p depend continuously on p ([**Er**]). In this note, we fix a finite set $B \subset F$ such that $\cup_n B^n = F$ and we consider probability measures in $\mathcal{P}(B)$, where $\mathcal{P}(B)$ is the set of probability measures p such that $p(x) > 0$ if, and only if, $x \in B$. The set $\mathcal{P}(B)$ is naturally identified with an open subset of the probabilities on B which is an open bounded convex domain in $\mathbb{R}^{|B|-1}$. We show:

Theorem 1.1. *Assume $F = \mathbb{F}_d$ is the free group with d generators, B is a finite subset of F such that $\cup_n B^n = F$. Then, with the above notation, the functions $p \mapsto h_p$ and $p \mapsto \ell_p$ are real analytic on $\mathcal{P}(B)$.*

Continuity of the entropy and of the linear drift is known for probabilities with first moment on a Gromov-hyperbolic group ([**EK**]). Also in the case when B is a set of free generators, there are formulas for the entropy and the linear drift which show that they are real analytic functions of the directing probability (see [**De2**] or imbed [**DM**] in the formulas (1) and (2) below). Similar formulas have been found for braid groups ([**M**]) and free products of finite groups or graphs ([**MM**], [**G1**], [**G2**]), but as soon as the set B is

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not reduced to the natural generating set, there is no direct formula for h_p or ℓ_p in terms of p.

The ratio h_p/ℓ_p has a geometric interpretation as the Hausdorff dimension D_p of the unique stationary measure for the action of F on the space ∂F of infinite reduced words. It follows from Theorem 1.1 that this dimension D_p is also real analytic in p, see below Corollary 2.1 for a more precise statement. Ruelle ([**R3**]) proved that the Hausdorff dimension of the Julia set of a rational function, as long as it is hyperbolic, depends real analytically of the parameters and our approach is inspired by [**R3**]. We first review properties of the random walk on F directed by a probability p. In particular, we can express h_p and ℓ_p in terms of the exit measure p^∞ of the random walk on the boundary ∂F (see [**Le**] and section 2 for background and notation). We then express this exit measure using thermodynamical formalism: if one views ∂F as a one-sided subshift of finite type, the exit measure p^∞ is the isolated eigenvector of maximal eigenvalue for a dual transfer operator $\mathcal{L}_p^*$ involving the Martin kernel of the random walk. Finally, from the description of the Martin kernel by Derriennic ([**De1**]), we prove that the mapping $p \mapsto \mathcal{L}_p$ is real analytic. The proof uses contractions in projective metric on complex cones ([**Ru**], [**Du1**]) and I want to thank Loïc Dubois for useful comments. Regularity of $p \mapsto p^\infty$ and Theorem 1.1 follow.

Our argument may apply to other similar settings. For instance, let $\pi : \mathbb{F}_d \to SO(k,1)$ be a faithful Schottky representation of the free group $\mathbb{F}_d$ as a convex cocompact group of $SO(k,1)$. Namely, $SO(k,1)$ is considered as a group of isometries of the hyperbolic space $\mathbb{H}^k$ and there are $2d$ disjoint open halfspaces H_a associated to the generators and their inverses in such a way that $\pi(a)$ sends the complement of $H_{a^{-1}}$ onto the closure of H_a in $\mathbb{H}^k$. Then, another natural asymptotic quantity is the Lyapunov exponent γ_p:

$$\gamma_p \ := \ \lim_n \frac{1}{n} \sum_{x \in F} p^{(n)}(x) \ln \|\pi(x)\|,$$

where $\|\cdot\|$ is some norm on matrices. Then,

Theorem 1.2. *Assume $\mathbb{F}_d$ is represented as a convex cocompact subgroup of $SO(k,1)$ as above, and B is a finite subset $B \subset F$ such that $\cup_n B^n = F$. Then the function $p \mapsto \gamma_p$ is a real analytic function on $\mathcal{P}(B)$.*

Analyticity of the exponent of an independent random product of matrices is known for *positive* matrices ([**R2**], [**P**], [**H**]). Here we show it for matrices in some *discrete* subgroup. It is possible that our approach yield similar results for more general discrete subgroups of $SO(k,1)$ or even for all Gromov-hyperbolic groups.

In the note, the letter C stands for a real number independent of the other variables, but which may vary from line to line. In the same way, the letter $\mathcal{O}_p$ stands for a neighborhood of $p \in \mathcal{P}(B)$ in $\mathbb{C}^B$ which may vary from line to line.

2. Convolutions of p

We recall in this section the properties of the convolutions $p^{(n)}$ of a finitely supported probability measure p on the free group $\mathbb{F}_d = F$. We follow the notation from [**Le**]. Any element of F has a unique reduced word representation in generators $\{a_1, \cdots, a_d, a_{-1}, \cdots, a_{-d}\}$. Set $\delta(x,x) = 0$ and, for $x \neq x'$, $\delta(x,x') = \exp -(x \wedge x')$, where $(x \wedge x')$ is the number of common letters at the beginning of the reduced word representations of x and x'. Then δ defines a metric on F and extends to the completion $F \cup \partial F$ with respect to δ. The boundary ∂F is a compact space which can be represented as the space of infinite reduced words. Then the distance between two distinct infinite reduced words ξ and ξ' is given by

$$\delta(\xi, \xi') \; = \; \exp -(\xi \wedge \xi'),$$

where $(\xi \wedge \xi')$ is the length of the initial common part of ξ and ξ'.

There is a natural continuous action of F over ∂F which extends the left action of F on itself: one concatenates the reduced word representation of $x \in F$ at the beginning of the infinite word ξ and one obtains a reduced word by making the necessary reductions. A probability measure μ on ∂F is called stationary if it satisfies

$$\mu \; = \; \sum_{x \in F} p(x) x_* \mu.$$

There is a unique stationary probability measure on ∂F, denoted p^∞ and the entropy h_p and the linear drift ℓ_p are given by the following formulae:

$$h_p \; = \; -\sum_{x \in F} \left(\int_{\partial F} \ln \frac{dx_*^{-1} p^\infty}{dp^\infty}(\xi) dp^\infty(\xi) \right) p(x), \tag{1}$$

$$\ell_p \; = \; \sum_{x \in F} \left(\int_{\partial F} \theta_\xi(x^{-1}) dp^\infty(\xi) \right) p(x), \tag{2}$$

where $\theta_\xi(x) = |x| - 2(\xi \wedge x) = \lim_{y \to \xi}(|x^{-1}y| - |y|)$ is the Busemann function.

Observe that in both expressions, the sum is a finite sum over $x \in B$. In the case of a finitely supported random walk on a general group, formula (1) holds, but with $(\partial F, p^\infty)$ replaced by the Poisson boundary of the random walk (see [**Fu**], [**Ka**]); formula (2) also holds, but with $(\partial F, p^\infty)$ replaced by some stationary measure on the Busemann boundary of the group ([**KL**]).

Recall that in the case of the free group the Hausdorff dimension of the measure p^∞ on $(\partial F, \delta)$ is given by h_p/ℓ_p ([**Le**], Theorem 4.15). So we have the following Corollary of Theorem 1.1:

Corollary 2.1. *Assume $F = \mathbb{F}_d$ is the free group with d generators, B is a finite subset of F such that $\cup_n B^n = F$. Then, with the above notation, the Hausdorff dimension of the stationary measure on $(\partial F, \delta)$ is a real analytic function of p in $\mathcal{P}(B)$.*

The Green function $G(x)$ associated to (F,p) is defined by

$$G(x) \ = \ \sum_{n=0}^{\infty} p^{(n)}(x)$$

(see Proposition 3.2 below for the convergence of the series). For $y \in F$, the Martin kernel K_y is defined by

$$K_y(x) = \frac{G(x^{-1}y)}{G(y)}.$$

Derriennic ([**De1**]) showed that $y_n \to \xi \in \partial F$ if, and only if, the Martin kernels K_{y_n} converge towards a function K_ξ called the Martin kernel at ξ. We have (see e.g. [**Le**] (3.11)):

$$\frac{dx_* p^\infty}{dp^\infty}(\xi) \ = \ K_\xi(x).$$

3. Random walk on F

The quantities introduced in Section 2 can be associated with the trajectories of a random walk on F. In this section, we recall the corresponding notation and properties. Let $\Omega = F^{\mathbb{N}}$ be the space of sequences of elements of F, M the product probability $p^{\mathbb{N}}$. The random walk is described by the probability $\mathbb{P}$ on the space of paths Ω, the image of M by the mapping:

$$(\omega_n)_{n\in\mathbb{Z}} \mapsto (X_n)_{n\geq 0}, \text{ where } X_0 = e \text{ and } X_n = X_{n-1}\omega_n \text{ for } n > 0.$$

In particular, the distribution of X_n is the convolution $p^{(n)}$. The notation p^∞ reflects the following

Theorem 3.1 (Furstenberg, see [**Le**], Theorem 1.12)**.** *There is a mapping $X_\infty : \Omega \to \partial F$ such that for M-a.e. ω,*

$$\lim_n X_n(\omega) \ = \ X_\infty(\omega).$$

The image measure p^∞ is the only stationary probability measure on ∂F.

For $x, y \in F$, let $u(x,y)$ be the probability of eventually reaching y when starting from x. By left invariance, $u(x,y) = u(e, x^{-1}y)$. Moreover, by the strong Markov property, $G(x) = u(e,x)G(e)$ so that we have:

$$K_y(x) = \frac{u(x,y)}{u(e,y)}. \tag{3}$$

By definition, we have $0 < u(x,y) \leq 1$. The number $u(x,y)$ is given by the sum of the probabilities of the paths going from x to y which do not visit y before arriving at y.

Proposition 3.2. *Let $p \in \mathcal{P}(B)$. There are numbers C and $\zeta, 0 < \zeta < 1$ and a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$ such that for all $q \in \mathcal{O}_p$, all $x \in F$ and all $n \geq 0$,*

$$|q|^{(n)}(x) \leq C\zeta^n.$$

Proof. Let $q \in \mathbb{C}^B$. Consider the convolution operator P_q in $\ell_2(F, \mathbb{R})$ defined by:

$$P_q f(x) = \sum_{y \in F} f(xy^{-1})|q|(y).$$

Derriennic and Guivarc'h ([**DG**]) showed that for $p \in \mathcal{P}(B)$, P_p has spectral radius smaller than one. In particular, there exists n_0 such that the operator norm of $P_p^{n_0}$ in $\ell_2(F)$ is smaller than one. Since B and B^{n_0} are finite, there is a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$ such that for all $q \in \mathcal{O}_p$, $\|P_q^{n_0}\|_2 < \lambda$ for some $\lambda < 1$ and $\|P_q^k\|_2 \leq C$ for $1 \leq k \leq n_0$. It follows that for all $q \in \mathcal{O}_p$, all $n \geq 0$,

$$\|P_q^n\|_2 \ \leq \ C\lambda^{[n/n_0]}.$$

In particular, for all $x \in F$, $|q|^{(n)}(x) = |[P_q^n \delta_e](x)| \leq |P_q^n \delta_e|_2 \leq C\lambda^{[n/n_0]}|\delta_e|_2 \leq C\lambda^{[n/n_0]}$. □

Fix $p \in \mathcal{P}(B)$. For $x \in F$, V a finite subset of F and $v \in V$, let $\alpha_x^V(v)$ be the probability that the first visit in V of the random walk starting from x occurs at v. We have $0 < \sum_{v \in V} \alpha_x^V(v) \leq 1$ and

Proposition 3.3. *Fix x, V and v. The mapping $p \mapsto \alpha_x^V(v)$ extends to an analytic function on a neighborhood of $\mathcal{P}(B)$ in $\mathbb{C}^B$.*

Proof. The number $\alpha_x^V(v)$ can be written as the sum of the probabilities $\alpha_x^{n,V}(v)$ of entering V at v in exactly n steps. The function $p \mapsto \alpha_x^{n,V}(v)$ is a polynomial of degree n on $\mathcal{P}(B)$:

$$\alpha_x^{n,V}(v) \ = \ \sum_{\mathcal{E}} q_{i_1} q_{i_2} \cdots q_{i_n},$$

where $\mathcal{E}$ is the set of paths $\{x, xi_1, xi_1 i_2, \cdots, xi_1 i_2 \cdots i_n = v\}$ of length n made of steps in B which start from x and enter V in v. By Proposition 3.2, there is a neighbourhood $\mathcal{O}_p$ of p in $\mathcal{P}(B)$ and numbers C, $\zeta, 0 < \zeta < 1$, such that for $q \in \mathcal{O}_p$ and for all $y \in F$,

$$|q|^{(n)}(y) \ \leq \ C\zeta^n.$$

It follows that for $q \in \mathcal{O}_p$,

$$|\alpha_x^{n,V}(v)| \leq C|q|^{(n)}(x^{-1}v) \leq C\zeta^n.$$

Therefore, $q \mapsto \alpha_x^V(v)$ is given locally by a uniformly converging series of polynomials, it is an analytic function on $\mathcal{O} := \cup_p \mathcal{O}_p$. □

4. Barriers and Hölder property of the Martin kernel

Set $r = \max\{|x|; x \in B\}$. A set V is called a barrier between x and y if $\delta(x,y) > r$ and if there exist two points z and z' of the geodesic between x and y, distinct from x and y such that $\delta(z,z') = r-1$ and V is the intersection of the two balls of radius $r-1$ centered at z and at z'. The basic geometric Lemma is the following:

Lemma 4.1 ([**De1**], Lemme 1)**.** *If x and y admit a barrier V, then every trajectory of the random walk starting from x and reaching y has to visit V before arriving at y.*

For V, W finite subsets of F, denote by A_V^W the matrix such that the row vectors are the $\alpha_v^W(w), w \in W$. In particular, if $W = \{y\}$, set u_V^y equal to the (column) vector

$$u_V^y = A_V^{\{y\}} = (\alpha_v^{\{y\}}(y))_{v\in V} = (u(v,y))_{v \in V}.$$

With this notation, Lemma 4.1 and the strong Markov property yield, if x and y admit V as a barrier:

$$u(x,y) \;=\; \sum \alpha_x^V(v)u(v,y) \;=\; \langle \alpha_x^V, u_V^y\rangle,$$

with the natural scalar product on $\mathbb{R}^V$. Then, Derriennic makes two observations: firstly, this formula iterates when one has k successive disjoint barriers between x and y and secondly there are only a finite number of possible matrices A_V^W when V and W are successive disjoint barriers with $\delta(V,W) = 1$. This gives the following formula for $u(x,y)$:

Lemma 4.2 ([**De1**], Lemme 2)**.** *Let $p \in \mathcal{P}(B)$. There are N square matrices with the same dimension $A_1, \cdots, A_N$, depending on p, such that for any $x, y \in F$, if $V_1, V_2, \cdots, V_k$ are disjoint successive barriers between x and y such that $\delta(V_i, V_{i+1}) = 1$ for $i = 1, \cdots, k-1$, there are $(k-1)$ indices $j_1, \cdots j_{k-1}$, depending only on the sequence V_i such that:*

$$u(x,y) = \langle \alpha_x^{V_1}, A_{j_1} \cdots A_{j_{k-1}} u_{V_k}^y\rangle. \tag{4}$$

By construction, the matrices A_j have nonnegative entries and satisfy $\sum_w A_j(v,w) \leq 1$. Moreover, we have the following properties:

Proposition 4.3 ([**De1**], Corollaire 1)**.** *Assume the set B contains the generators and their inverses, then for each $p \in \mathcal{P}(B)$, for each $j = 1, \cdots, N$, the matrix A_j has all its* 0 *entries in full columns.*

From the proof of proposition 4.3, if the set B contains the generators and their inverses and $A_j = A_{V_j}^{V_{j+1}}$, columns of 0's correspond to the subset W_{j+1} of points in V_{j+1} which cannot be entry points from paths starting in V_j. In particular, they depend only of the geometry of B and are the same for all $p \in \mathcal{P}(B)$.

We may – and we shall from now on – assume that the set B contains the generators and their inverses. Indeed, since $h_{p^{(k)}} = kh_p$ and $\ell_{p^{(k)}} = k\ell_p$, we can replace in Theorem 1.1 the probability p by a convolution of order high enough that the generators and their inverses have positive probability. Then, by Proposition 4.3 the matrices $A_j(q)$ have the same columns of zeros for all $q \in \mathcal{P}(B)$. Moreover,

Proposition 4.4. *For each $j = 1, \cdots, N$, the mapping $p \mapsto A_j$ extends to analytic function on a neighborhood of $\mathcal{P}(B)$ in $\mathbb{C}^B$ into the set of complex matrices with the same configuration of zeros as A_j.*

Proof. The proof is completely analogous to the proof of Proposition 3.3; one may have to take a smaller neighborhood for the sake of avoiding introducing new zeros. $\square$

We are interested in the function $\Phi : \partial F \to \mathbb{R}, \Phi(\xi) = -\ln K_\xi(\xi_1)$. By (3), (4) and Deriennic's Theorem, we have:

$$\begin{aligned}\Phi(\xi) &= -\ln \lim_{n\to\infty} K_{\xi_1\xi_2\cdots\xi_n}(\xi_1)\\ &= -\ln \lim_{n\to\infty} \frac{u(\xi_1, \xi_1\xi_2\cdots\xi_n)}{u(e, \xi_1\xi_2\cdots\xi_n)}\\ &= -\ln \lim_{k\to\infty} \frac{\langle \alpha_{\xi_1}^{V_1(\xi)}, A_{j_1}(\xi)\cdots A_{j_{k-1}}(\xi) u_{V_k(\xi)}^{y_k}\rangle}{\langle \alpha_{e}^{V_1(\xi)}, A_{j_1}(\xi)\cdots A_{j_{k-1}}(\xi) u_{V_k(\xi)}^{y_k}\rangle},\end{aligned}$$

where $A_{j_s}(\xi) = A_{V_s(\xi)}^{V_{s+1}(\xi)}$, the $V_s(\xi)$ are successive disjoint barriers between ξ_1 and ξ with $\delta(V_s(\xi), V_{s+1}(\xi)) = 1$ for all $s > 1$, $\delta(\xi_1, V_1) = 1$ and y_k is the closest point beyond V_k on the geodesic from ξ_1 to ξ.

Define on the nonnegative convex cone C_0 in $\mathbb{R}^m$ the projective distance between half lines as

$$\vartheta(f,g) := |\ln[f,g,h,h']|,$$

where h, h' are the intersections of the boundaries of the cone with the plane (f,g) and $[f,g,h,h']$ is the cross ratio of the four directions in the same plane. Represent the space of directions as the sector of the unit sphere $D = C_0 \cup S^{m-1}$; then, ϑ defines a distance on D. Let A be a $m \times m$ matrix with nonpositive entries and let $T : D \to D$ be the projective action of A. Then, by [**Bi**]:

$$\vartheta(Tf, Tg) \le \beta\vartheta(f,g), \text{ where } \beta = \tanh\left(\frac{1}{4} \text{ Diam } T(D)\right). \tag{5}$$

When A_j is one of the matrices of Lemma 4.2, it acts on $\mathbb{R}^V$ and the image $T_j(D)$ has finite diameter, so that $\beta_j := \tanh\left(\frac{1}{4} \text{ Diam } T_j(D)\right) < 1$. Set $\beta_0 := \max_{j=1,\cdots,N} \beta_j$. Then, $\beta_0 < 1$.

Set $f_k(\xi) := \dfrac{u_{V_k(\xi)}^{y_k}}{\|u_{V_k(\xi)}^{y_k}\|}, \alpha(\xi) := \alpha_e^{V_1(\xi)}, \alpha_1(\xi) := \alpha_{\xi_1}^{V_1(\xi)}$. For all ξ, $f_k(\xi) \in D$ and $\alpha(\xi), \alpha_1(\xi) \in C_0 - \{0\}$. The above formula for $\Phi(\xi)$ becomes:

$$\Phi(\xi) = -\ln \lim_{k\to\infty} \frac{\langle \alpha_1(\xi), T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi) f_k(\xi)\rangle}{\langle \alpha(\xi), T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi) f_k(\xi)\rangle}. \tag{6}$$

Proposition 4.5. *Fix $p \in \mathcal{P}$. The function $\xi \mapsto \Phi(\xi)$ is Hölder continuous on ∂F.*

Proof. Let ξ, ξ' be two points of ∂F with $\delta(\xi, \xi') \leq \exp(-((n+1)r+1))$. The points ξ and ξ' have the same first $(n+1)r+1$ coordinates. In particular, $V_s(\xi) = V_s(\xi')$ for $1 \leq s \leq n$. By using (6), we see that $\Phi(\xi') - \Phi(\xi)$ is given by the limit, as k goes to infinity, of

$$\ln \frac{\langle \alpha_1(\xi), T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_k(\xi) \rangle}{\langle \alpha_1(\xi'), T_{j_1}(\xi') \cdots T_{j_{k-1}}(\xi') f_k(\xi') \rangle} \frac{\langle \alpha(\xi'), T_{j_1}(\xi') \cdots T_{j_{k-1}}(\xi') f_k(\xi') \rangle}{\langle \alpha(\xi), T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_k(\xi) \rangle}.$$

We have $\alpha_1(\xi) = \alpha_1(\xi') =: \alpha_1, \alpha(\xi) = \alpha(\xi') =: \alpha$ and $T_{j_s}(\xi) = T_{j_s}(\xi') =: T_{j_s}$ for $s = 1, \cdots, n$. Moreover, for any $f, f' \in D$,

$$\vartheta\left(T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f, T_{j_1}(\xi') \cdots T_{j_{k-1}}(\xi') f'\right) \ = \ \vartheta\left(T_{j_1} \cdots T_{j_{n-1}} g_k, T_{j_1} \cdots T_{j_{n-1}} g'_k\right)$$

for $g_k = T_{j_n} T_{j_{n+1}}(\xi) \cdots T_{j_{k-1}}(\xi) f, g'_k = T_{j_n} T_{j_{n+1}}(\xi') \cdots T_{j_{k-1}}(\xi') f'$.

We have $\vartheta(g_k, g'_k) \leq$ Diam $T_{j_n} D < \infty$ and, by repeated application of (5),

$$\vartheta\left(T_{j_1} \cdots T_{j_{n-1}} g_k, T_{j_1} \cdots T_{j_{n-1}} g'_k\right) \ \leq \ \beta_0^{n-1} \vartheta\left(g_k, g'_k\right) \ \leq C \beta_0^n. \tag{7}$$

Using all the above notation, we get

$$\Phi(\xi) - \Phi(\xi') = \ln \lim_k \frac{\langle \alpha_1, T_{j_1} \cdots T_{j_{n-1}} g'_k \rangle}{\langle \alpha_1, T_{j_1} \cdots T_{j_{n-1}} g_k \rangle} \frac{\langle \alpha, T_{j_1} \cdots T_{j_{n-1}} g_k \rangle}{\langle \alpha, T_{j_1} \cdots T_{j_{n-1}} g'_k \rangle}. \tag{8}$$

As ξ varies, α and α_1 belong to a finite family of vectors of $C_0 - \{0\}$. It then follows from (7) that, as soon as $\delta(\xi, \xi') \leq \exp(-((n+1)r+1))$, $|\Phi(\xi) - \Phi(\xi')| \leq C \beta_0^n$. □

Let us choose $\beta, \beta_0^{1/r} < \beta < 1$, and consider the space Γ_β of functions ϕ on ∂F such that there is a constant C_β with the property that, if the points ξ and ξ' have the same first n coordinates, then $|\phi(\xi) - \phi(\xi')| < C_\beta \beta^n$. For $\phi \in \Gamma_\beta$, denote $\|\phi\|_\beta$ the best constant C_β in this definition. The space Γ_β is a Banach space for the norm $\|\phi\| := \|\phi\|_\beta + \max_{\partial F} |\phi|$. Proposition 4.5 says that for $p \in \mathcal{P}(B)$, the function $\Phi_p(\xi) = -\ln K_\xi(\xi_1)$ belongs to Γ_β.

5. Regularity of the Martin kernel

We want to extend the mapping $p \mapsto \Phi_p$ to a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$. Firstly, we redefine Γ_γ as the space of complex functions ϕ on ∂F such that there is a constant C_γ with the property that, for all $n \geq 0$, if the points ξ and ξ' have the same first n coordinates, then $|\phi(\xi) - \phi(\xi')| < C_\gamma \gamma^n$. The space Γ_γ is a complex Banach space for the norm $\|\phi\| := \|\phi\|_\gamma + \max_{\partial F} |\phi|$, where $\|\phi\|_\gamma$ the best possible constant C_γ. In this section, we find a neighborhood $\mathcal{O}_p$ and a $\gamma = \gamma(p), 0 < \gamma < 1$, such that formula (6) makes sense on $\mathcal{O}_p$ and defines a function in Γ_γ.

In recent papers, Rugh ([**Ru**]) and Dubois ([**Du1**]) show how to extend (5) to the complex setting. In a complex Banach space X, they define a $\mathbb{C}$-cone as a subset invariant by multiplication by $\mathbb{C}$, different from $\{0\}$ and not containing any complex 2-dimensional subspace in its closure. A $\mathbb{C}$-cone $\mathcal{C}$ is called linearly convex if each point in the complement of $\mathcal{C}$ is contain in a complex hyperplane not intersecting $\mathcal{C}$. Let $K < +\infty$. A $\mathbb{C}$-cone $\mathcal{C}$ is called K-regular if it has some interior and if, for each vector space P of complex dimension 2, there is some nonzero linear form $m \in X^*$ such that, for all $u \in \mathcal{C} \cap P$,

$$\|m\|\|u\| \ \leq \ K|\langle m, u\rangle|.$$

Let $\mathcal{C}$ be a linearly convex $\mathbb{C}$-cone. A projective distance $\vartheta_{\mathcal{C}}$ on $(\mathcal{C} - \{0\}) \times (\mathcal{C} - \{0\})$ is defined as follows ([**Du1**], Section 2): if f and g are colinear, set $\vartheta_{\mathcal{C}}(x, y) = 0$; otherwise, consider the following set $E(f, g)$:

$$E(f, g) \ := \ \{z, z \in \mathbb{C}, zf - g \notin \mathcal{C}\},$$

and then define

$$\vartheta_{\mathcal{C}}(f, g) = \ln \frac{b}{a}, \quad \text{where } b = \sup |E(f, g)| \in (0, +\infty], a = \inf |E(f, g)| \in [0, +\infty).$$

Proposition 5.1 ([**Du1**], Theorem 2.7)**.** *Let X_1, X_2 be complex Banach spaces, and let $\mathcal{C}_1 \subset X_1, \mathcal{C}_2 \subset X_2$ be complex cones. Let $A : X_1 \to X_2$ be a linear map with $A(\mathcal{C}_1 - \{0\}) \subset (\mathcal{C}_2 - \{0\})$ and assume that*

$$\Delta := \sup_{f,g \in (\mathcal{C}_1 - \{0\})} \vartheta_{\mathcal{C}_2}(Af, Ag) < +\infty.$$

Then, for all $f, g \in \mathcal{C}_1$,

$$\vartheta_{\mathcal{C}_2}(Af, Ag) \ \leq \ \tanh\left(\frac{\Delta}{4}\right)\vartheta_{\mathcal{C}_1}(f, g). \tag{9}$$

Proposition 5.2 ([**Du1**], Lemma 2.6)**.** *Let $\mathcal{C}$ be a K-regular, linearly convex $\mathbb{C}$-cone and let $f \sim g$ if, and only if, there is $\lambda, \lambda \neq 0$ such that $\lambda f = g$. Then $\vartheta_{\mathcal{C}}$ defines a complete projective metric on $\mathcal{C}/\sim$. Moreover, if $f, g \in \mathcal{C}, \|f\| = \|g\| = 1$, then there is a complex number ρ of modulus 1, $\rho = \rho(f, g)$, such that*

$$\|\rho f - g\| \ \leq \ K\vartheta_{\mathcal{C}}(f, g). \tag{10}$$

Proposition 5.3 ([**Ru**], Corollary 5.6, [**Du1**], Remark 3.6)**.** *For $m \geq 1$, the set*

$$\mathbb{C}^m_+ = \{u \in \mathbb{C}^m : \mathfrak{Re}(u_i \overline{u_j}) \geq 0, \forall i, j\} = \{u \in \mathbb{C}^m : |u_i + u_j| \geq |u_i - u_j|, \forall i, j\}$$

is a regular linearly convex $\mathbb{C}$-cone. The inclusion

$$\pi : (C_0 - \{0\}, \vartheta) \longrightarrow (\mathbb{C}^m_+ - \{0\}, \vartheta_{\mathbb{C}^m_+})$$

is an isometric embedding.

Moreover, [**Du1**] studies and characterizes the $m \times m$ matrices which preserve $\mathbb{C}^m_+$. We need the following properties. Let A be a $m \times m$ matrix with all 0 entries in m' full columns and $\lambda_1, \cdots, \lambda_m$ the $(m-m')$-row vectors made up of the nonzeros entries of the row vectors of A. Set:

$$\delta_{k,l} := \vartheta_{\mathbb{C}^{m-m'}_+}(\lambda_k, \lambda_l), \quad \Delta_{k,l} := \mathrm{Diam}_{\mathrm{RHP}}\Big\{\frac{\langle \lambda_k, x\rangle}{\langle \lambda_l, x\rangle}; x \in (\mathbb{C}^{m-m'}_+)^*, x \neq 0\Big\},$$

where $\mathrm{Diam}_{\mathrm{RHP}}$ denotes the diameter with respect to the Poincaré metric of the right half-plane. Observe that $\mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^m - \{0\})) = \mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^{m-m'} - \{0\}))$. Then, we have ([**Du1**], Proposition 3.5):

$$\mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^m - \{0\})) \;\le\; \max_{k,l} \delta_{k,l} + 2\max_{k,l} \Delta_{k,l} \;\le\; 3\mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^m - \{0\})). \tag{11}$$

From the proof of Proposition 3.5 in [**Du1**], in particular from equation (3.12), it also follows that for a real matrix A:

$$\mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^m - \{0\})) \;\le\; 3\mathrm{Diam}_{\vartheta}(A(\mathbb{R}_+^m - \{0\})).$$

Fix $p \in \mathcal{P}(B)$. We choose $\gamma = \gamma(p) < 1$ such that

$$9(\tanh)^{-1}\beta_0 < (\tanh)^{-1}(\gamma^{2r}).$$

Then for the real matrices $A = A_1(p), \cdots, A_N(p)$,

$$3\mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^m - \{0\})) \le 9\mathrm{Diam}\ _{\vartheta}(A(\mathbb{R}_+^m - \{0\})) \le 36(\tanh)^{-1}\beta_0 < 4(\tanh)^{-1}(\gamma^{2r}). \tag{12}$$

Proposition 5.4. *Fix $p \in \mathcal{P}(B)$. There is a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$ such that the mapping $p \mapsto \Phi_p$ extends to an analytic mapping from $\mathcal{O}_p$ into $\Gamma_{\gamma(p)}$.*

Proof. We first extend $A_j, j = 1, \cdots, N$ analytically on a neighborhood $\mathcal{O}_p$ by Proposition 4.4. Set $S = S^{2m-1} = \{f; f \in \mathbb{C}_+^m, \|f\| = 1\}$. For each $A_j(q), j = 1, \cdots N, q \in \mathcal{O}_p$ and each $f \in S$ such that $A_j(q)f \neq 0$, we define again $T_j(q)f$ by:

$$T_j(q)f \;=\; \frac{A_j(q)f}{\|A_j(q)f\|}.$$

For $p \in \mathcal{P}(B)$, the function Φ_p is given by the limit from formula (6):

$$\Phi_p(\xi) = -\ln \lim_{k\to\infty} \frac{\langle \alpha_1(\xi), T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi) f_0\rangle}{\langle \alpha(\xi), T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi) f_0\rangle},$$

where $f_0 \in S$ the column vector $\{1/\sqrt{|B|}, \cdots, 1/\sqrt{|B|}\}$: we use the fact that the limit of $T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi)f$ does not depend on the initial point f.

We have to show that this limit extends on some neighborhood $\mathcal{O}_p$ of p to an analytic function into Γ_γ. Set

$$\Phi_{p,k}(\xi) \;:=\; -\ln \frac{\langle \alpha_1(\xi), A_{j_1}(\xi)\cdots A_{j_{k-1}}(\xi) f_0\rangle}{\langle \alpha(\xi), A_{j_1}(\xi)\cdots A_{j_{k-1}}(\xi) f_0\rangle}.$$

We are going to find $\mathcal{O}_p$ and k_0 such that, for $k \ge k_0$, the functions $\Phi_{p,k}(\xi)$ extend to analytic functions from $\mathcal{O}_p$ into Γ_γ and, as $k \to \infty$, the functions $\Phi_{p,k}(\xi)$ converge in Γ_γ uniformly on $\mathcal{O}_p$.

The functions $q \mapsto \langle \alpha_1(\xi), A_{j_1}(\xi)\cdots A_{j_{k-1}}(\xi) f_0\rangle, q \mapsto \langle \alpha(\xi), A_{j_1}(\xi)\cdots A_{j_{k-1}}(\xi) f_0\rangle$ are polynomials in q and depend only on a finite number of coordinates of ξ. Therefore, if we can find a neighborhood $\mathcal{O}_p$ and a k such that these two functions do not vanish, then $\Phi_{p,k}$ extends to an analytic function from $\mathcal{O}_p$ to Γ_γ.

Step1: Contraction

By (11), (12) and Proposition 4.4, we can choose a neighborhood $\mathcal{O}_p$ such that for $q \in \mathcal{O}_p$, the diameter Δ of $A_j(q)\mathbb{C}_+^m$ is smaller than $4(\tanh)^{-1}(\gamma^{2r})$ for all $j = 1, \cdots, N$.[1] The set $\mathcal{D} := S \cap \big(\cup_j A_j(p)\mathbb{C}_+^m\big)$ is compactly contained in the interior of S. We choose a smaller neighborhood $\mathcal{O}_p$ such that, if $q \in \mathcal{O}_p$,

$$\Delta < 4(\tanh)^{-1}(\gamma^{2r}) \quad \text{and} \quad 0 \notin A_j(\mathcal{D} \cup \{f_0\}) \;\text{ for } j = 1, \cdots, N.$$

For $q \in \mathcal{O}_p$, the projective images $T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_0$ are all defined and we have, by repeated application of (9)

$$\vartheta_{\mathcal{C}}\left(T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_0, T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_{k,k'}(\xi)\right) \;\le\; \gamma^{2(k-1)r} \vartheta(f_0, f_{k,k'}(\xi)),$$

where $k' > k$ and $f_{k,k'}(\xi) := T_{j_k}(\xi) \cdots T_{j_{k'-1}}(\xi) f_0$. The $f_{k,k'}(\xi)$ are all in $\mathcal{D}$. Then, $\vartheta_{\mathcal{C}}(f_0, f_{k,k'}(\xi)) \le C$ for all $\xi \in \partial F$, all $k, k' \ge 1$. Set

$$g = T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_0, \quad g' = T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_{k,k'}(\xi).$$

For all $\xi \in \partial F$, all $k, k' \ge 1$, consider the number $\rho(\xi, k, k')$ associated by Proposition 5.2 to g and g'. We have, by (10):

$$|\rho(\xi, k, k')| = 1 \quad \text{and} \quad \|\rho(\xi, k, k') g - g'\| \;\le\; KC\gamma^{2kr}.$$

Since $\alpha(p, \xi)$ and $\alpha_1(p, \xi)$ take finite many values, it follows that

$$\begin{aligned} & |\langle \alpha(p,\xi), g\rangle \langle \alpha_1(p,\xi), g'\rangle - \langle \alpha(p,\xi), g'\rangle \langle \alpha_1(p,\xi), g\rangle| \\ = \;& |\langle \alpha(p,\xi), \rho(\xi,k,k') g\rangle \langle \alpha_1(p,\xi), g'\rangle - \langle \alpha(p,\xi), g'\rangle \langle \alpha_1(p,\xi), \rho(\xi,k,k') g\rangle| \\ \le \;& KC\gamma^{2kr}. \end{aligned}$$

Since g and g' are in the compact set $\mathcal{D} \cup \{f_0\}$, we can, by Proposition 3.3, choose a neighborhood $\mathcal{O}_p$ such that, for all $q \in \mathcal{O}_p$, all $\xi \in \partial F$, all $k < k'$

$$\begin{aligned} & \big|\langle \alpha(\xi), T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_0\rangle \langle \alpha_1(\xi), T_{j_1}(\xi) \cdots T_{j_{k'-1}}(\xi) f_0\rangle \\ & - \langle \alpha(\xi), T_{j_1}(\xi) \cdots T_{j_{k'-1}}(\xi) f_0\rangle \langle \alpha_1(\xi), T_{j_1}(\xi) \cdots T_{j_{k-1}}(\xi) f_0\rangle\big| \;\le\; KC\gamma^{2kr}. \end{aligned}$$

Step 2: The $\Phi_{q,k}$ extend

Recall that D is the set of unit vectors in the positive quadrant. For $g, g' \in \cup_j T_j(p)(D) \cup \{f_0\}$, $\langle \alpha(p,\xi), g\rangle \langle \alpha_1(p,\xi), g'\rangle$ is real positive and bounded away from 0 uniformly in ξ, g and g'. Recall the isometric inclusion $\pi : D \to S$ of Proposition 5.3. There is a neighborhood $\mathcal{C}_0$ of $\pi\big(\cup_j T_j(p)(D) \cup \{f_0\}\big)$ in S and $\delta > 0$ such that for $g, g' \in \mathcal{C}_0$, $\big|\langle \alpha(p,\xi), g\rangle \langle \alpha_1(p,\xi), g'\rangle\big| > \delta$. Of course, we can take $\mathcal{C}_0$ invariant by multiplication by all z with $|z| = 1$. Then, there exists $\varepsilon > 0$ such that if $\vartheta_{C_+^m}\big(g, \pi(\cup_j T_j(p)(D) \cup \{f_0\})\big) < \varepsilon, \vartheta_{C_+^m}\big(g', \pi(\cup_j T_j(p)(D) \cup \{f_0\})\big) < \varepsilon$, then $\big|\langle \alpha(p,\xi), g\rangle \langle \alpha_1(p,\xi), g'\rangle\big| > \delta/2$.

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[1]One can also use directly [**Du2**], Theorem 4.5.

For $q \in \mathcal{O}_p$ and $k_0 > 1 + \ln(\varepsilon/2)/2r\ln\gamma$, the $\vartheta_{\mathbb{C}_+^m}$-diameter of each one of the sets $T_{j_1}(q,\xi)\cdots T_{j_{k_0-1}}(q,\xi)S$ is smaller than $\varepsilon/2$, for all ξ. As ξ varies, there is only a finite number of mappings $T_{j_1}(q,\xi)\cdots T_{j_{k_0-1}}(q,\xi)$. By continuity of $q \mapsto T_j$ (where the T_js now are considered as mappings from $\mathcal{C}/\sim$ into itself), there is a neighborhood $\mathcal{O}_p$ such that for $q \in \mathcal{O}_p$, the Hausdorff distance between $T_{j_1}(q,\xi)\cdots T_{j_{k_0-1}}(q,\xi)S/\sim$ and $T_{j_1}(p,\xi)\cdots T_{j_{k_0-1}}(p,\xi)S/\sim$ is smaller than $\varepsilon/2$. It follows that if $q \in \mathcal{O}_p$, and g,g' are in the same $T_{j_1}(q,\xi)\cdots T_{j_{k_0-1}}(q,\xi)S$ for some ξ, then

$$\big|\langle\alpha(p,\xi),g\rangle\langle\alpha_1(p,\xi),g'\rangle\big| > \delta/2.$$

By taking a possibly smaller $\mathcal{O}_p$, we have that if $q \in \mathcal{O}_p$, and g,g' are in the same $T_{j_1}(q,\xi)\cdots T_{j_{k_0-1}}(q,\xi)S$ for some ξ, then

$$\big|\langle\alpha(q,\xi),g\rangle\langle\alpha_1(q,\xi),g'\rangle\big| > \delta/4.$$

In particular this last expression does not vanish and $\Phi_{q,k}$ is an analytic function on $\mathcal{O}_p$ for $k \geq k_0$.

Step3: The $\Phi_{q,k}$ converge uniformly on ∂F

Take a neighborhood $\mathcal{O}_p$ and k_0 such that for $q \in \mathcal{O}_p$ the conclusions of Steps 1 and 2 hold. We claim that for all $\varepsilon > 0$, there is k_1 such that for $k,k' \geq k_1$, $q \in \mathcal{O}_p$, $\max_\xi|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)| < \varepsilon$. Suppose $k_1 > k_0$. We have to estimate

$$\max_\xi\Big|\ln\frac{\langle\alpha(\xi),T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi)f_0\rangle}{\langle\alpha(\xi),T_{j_1}(\xi)\cdots T_{j_{k'-1}}(\xi)f_0\rangle}\frac{\langle\alpha_1(\xi),T_{j_1}(\xi)\cdots T_{j_{k'-1}}(\xi)f_0\rangle}{\langle\alpha_1(\xi),T_{j_1}(\xi)\cdots T_{j_{k-1}}(\xi)f_0\rangle}\Big|.$$

By the conclusions of Steps 1 and 2, this quantity is smaller that $C\max\{\gamma^{2kr},\gamma^{2k'r}\}$. This is smaller than ε if k_1 is large enough.

Step 4: The $\Phi_{q,k}$ converge in norm $\|.\|_{\gamma(p)}$

With the same $\mathcal{O}_p, k_0$, we now claim that for all $\varepsilon > 0$, there is $k_2 = \max\{k_0, \ln\gamma/r\ln\varepsilon\}$ such that for $k,k' \geq k_2$ and $q \in \mathcal{O}_p$, $\|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)\|_\gamma < \varepsilon$. Let ξ,ξ' be two points of ∂F with $\delta(\xi,\xi') \leq \exp(-((n+1)r+1))$. We want to show that there is a constant C independent on n, such that, for all $q \in \mathcal{O}_p$, all $k,k' \geq k_2$:

$$|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi) - \Phi_{q,k}(\xi') + \Phi_{q,k'}(\xi')| \;\leq\; C\gamma^{(n+1)r+1}\varepsilon.$$

Since $k,k' \geq k_0$, the difference $\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)$ is given by:

$$\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi) = \ln\frac{\langle\alpha_1,T_{j_1}\cdots T_{j_{k'-1}}f_0\rangle}{\langle\alpha_1,T_{j_1}\cdots T_{j_{k-1}}f_0\rangle}\frac{\langle\alpha,T_{j_1}\cdots T_{j_{k-1}}f_0\rangle}{\langle\alpha,T_{j_1}\cdots T_{j_{k'-1}}f_0\rangle}.$$

For $k,k' \leq n+1$, $\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi) = \Phi_{q,k}(\xi') - \Phi_{q,k'}(\xi')$, and there is nothing to prove.

Assume $k' > k \geq n+1$, Step 3 shows that both $|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)|$ and $|\Phi_{q,k}(\xi') - \Phi_{q,k'}(\xi')|$ are smaller than $C\gamma^{2kr} \leq C\gamma^{nr}\gamma^{kr} \leq C\gamma^{nr}\varepsilon$.

The remaining case, when $k_0 \leq k \leq n+1 \leq k'$, clearly follows from the other two and this shows Step 4.

Finally we have that the functions $\Phi_{p,k}$ are analytic and converge uniformly in Γ_γ on a neighborhood $\mathcal{O}_p$ of p. The limit is an analytic function on $\mathcal{O}_p$.

□

6. Proof of Theorem 1.1

In this section, we consider ∂F as a subshift of finite type and let τ be the shift transformation on ∂F:

$$\tau\xi = \eta_1\eta_2\cdots \quad \text{with} \quad \eta_n = \xi_{n+1}.$$

For $\gamma < 1$ and $\phi \in \Gamma_\gamma$ with real values, we define the transfer operator $\mathcal{L}_\phi$ on Γ_γ by

$$\mathcal{L}_\phi\psi(\xi) := \sum_{\eta\in\tau^{-1}\xi} e^{\phi(\eta)}\psi(\eta).$$

Then, $\mathcal{L}_\phi$ is a bounded operator in Γ_γ and, by Ruelle's transfer operator theorem (see e.g. [**Bo**]), there exists a number $P(\phi)$, a positive function $h_\phi \in \Gamma_\gamma$ and an unique linear functional ν_ϕ on Γ_γ such that:

$$\mathcal{L}_\phi h_\phi = e^{P(\phi)}h_\phi, \quad \mathcal{L}_\phi^*\nu_\phi = e^{P(\phi)}\nu_\phi \quad \text{and} \quad \nu_\phi(1) = 1.$$

The functional ν_ϕ extends to probability measure on ∂F and is the only eigenvector of $\mathcal{L}_\phi^*$ with that property. Moreover, $\phi \mapsto \mathcal{L}_\phi$ is a real analytic map from Γ_γ to the space of linear operators on Γ_γ ([**R1**], page 91). Consequently, the mapping $\phi \mapsto \nu_\phi$ is real analytic from Γ_γ into the dual space Γ_γ^* (see e.g. [**Co**], Corollary 4.6). By Proposition 5.4, the mapping $p \mapsto \nu_{\Phi_p}$ is real analytic from a neighborhood of p in $\mathcal{P}(B)$ into the space $\Gamma_{\gamma(p)}^*$.

The main observation is that, for all $p \in \mathcal{P}(B)$, $\mathcal{L}_{\Phi_p}^* p^\infty = p^\infty$; this implies that $P(\Phi_p) = 0$ and that the distribution ν_{Φ_p} is the restriction of the measure p^∞ to any Γ_γ such that $\Phi_p \in \Gamma_\gamma$. Indeed, we have:

$$\frac{d\tau_* p^\infty}{dp^\infty}(\xi) = \frac{d(\xi_1)_* p^\infty}{dp^\infty} = K_\xi(\xi_1) = e^{\Phi_p(\xi)}$$

so that, for all continuous ψ:

$$\int (\mathcal{L}_{\Phi_p}\psi)dp^\infty = \sum_a \int_{a\xi,\xi_1\neq a^{-1}} \frac{dp^\infty(a\xi)}{dp^\infty(\xi)}\psi(a\xi)dp^\infty(\xi) = \int \psi dp^\infty.$$

Recall the equations (1) and (2) for h_p and ℓ_p. ℓ_p is given by a finite sum (in x) of integrals with respect to p^∞ of the functions $\xi \mapsto \theta_\xi(x)$. Since these functions only depend on a finite number of coordinates in ∂F, they belong to Γ_γ for all $\gamma < 1$. Since $p \mapsto \nu_{\Phi_p}$ is real analytic from a neighborhood of p into $\Gamma_{\gamma(p)}^*$, $p \mapsto \ell_p$ is real analytic on a neighborhood of p. Since this is true for all $p \in \mathcal{P}(B)$, the function $p \mapsto \ell_p$ is real analytic on $\mathcal{P}(B)$.

The argument is the same for h_p, since the function $\ln \frac{dx_*^{-1}p^\infty}{dp^\infty}(\xi) = \ln K_\xi(x^{-1}) \in \Gamma_\gamma$ for all x and for all $\gamma, \beta < \gamma < 1$ and the mappings $p \mapsto \ln K_\xi(x^{-1})$ are real analytic from a neighborhood of p into $\Gamma_{\gamma(p)}$. Indeed, $\ln K_\xi(\xi_1) \in \Gamma_\beta$ by Proposition 4.5 and $p \mapsto \ln K_\xi(\xi_1)$ is real analytic into $\Gamma_{\gamma(p)}$ by Proposition 5.4. Moreover, if a is a generator different from

ξ_1, $\ln K_\xi(a) = -\ln K_{a^{-1}\xi}(a^{-1})$ also lies in Γ_β and $p \mapsto \ln K_\xi(a)$ is also real analytic into $\Gamma_{\gamma(p)}$. For a general $x \in F$, $x = a_1 \cdots a_t$, write

$$K_\xi(x^{-1}) \;=\; K_\xi(a_t^{-1} \cdots a_1^{-1}) \;=\; K_\xi(a_t^{-1}) K_{a_t\xi}(a_{t-1}^{-1}) \cdots K_{a_2\cdots a_t\xi}(a_1^{-1}).$$

This completes the proof of Theorem 1.1. For the proof of Theorem 1.2, fix an origin $o \in \mathbb{H}^k$. Then, $\pi(F)o$ accumulates to the boundary of $\mathbb{H}^k$ in a Cantor set Λ called the limit set of $\pi(F)$. The mapping $\pi_o : F \to \mathbb{H}^n, \pi_o(x) = x.o$ extends to a Hölder continuous mapping π_o from ∂F to the limit set Λ of $\pi(F)$. We can express the exponent γ_p as:

$$\gamma_p \;=\; \lim_n \frac{1}{2n} \sum_{x \in F} d(o, \pi_o(x)) p^{(n)}(x),$$

where the distance d is the hyperbolic distance in $\mathbb{H}^k$. We obtain, in the same way as for formula (2),

$$\begin{aligned}
\gamma_p \quad &= \quad \frac{1}{2} \sum_{x \in F} \left(\int_\Lambda \Theta_\zeta(\pi_o(x^{-1})) d((\pi_o)_* p^\infty)(\zeta) \right) p(x) \\
&= \quad \frac{1}{2} \sum_{x \in F} \left(\int_{\partial F} \Theta_{\pi_o(\xi)}(\pi_o(x^{-1})) d(p^\infty)(\xi) \right) p(x),
\end{aligned}$$

where Θ_ζ is now the Busemann function of $\mathbb{H}^k$: $\Theta_\zeta(z) := \lim_{w \to \zeta} d(w,z) - d(w,o)$. Since, for all $x \in F$, the function $\xi \mapsto \Theta_{\pi_o(\xi)}(\pi_o(x))$ is a ρ-Hölder continuous function for some fixed ρ, we deduce as above that $p \mapsto \gamma_p$ is real analytic on $\mathcal{P}(B)$.

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